COMMENSURABILITY AND QI CLASSIFICATION OF FREE PRODUCTS OF FINITELY GENERATED ABELIAN GROUPS

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The following gives the complete commensurability and quasi-isometry classification of free products of finitely generated abelian groups. The quasi-isometry classification is a special case of Papasoglu and Whyte [4].

Theorem 1. *Let G_i be a free product of a finite set S_i of finitely generated abelian groups for $i = 1, 2$. Then the following are equivalent*

1. *The sets of ranks ≥ 2 of groups in S_1 and S_2 are equal (the* rank *of a finitely generated group is the rank of its free abelian part);*
2. *G_1 and G_2 are commensurable.*
3. *G_1 and G_2 are quasi-isometric.*

Proof. The main step is to show (1) implies (2). By going to finite index subgroups of G_1 and G_2 we can assume the groups in S_1 and S_2 are free abelian (take the kernel of the map of G_i to a product of finite quotients of the groups in S_i by torsion free normal subgroups, or see the lemma below for a more general statement). Let $n_1 = 1$ and let $n_2, \dots, n_k$ be the ranks ≥ 2 of the groups in S_1 and in S_2. Let r_i and s_i be the number of rank n_i groups in S_1 and S_2 respectively. We identify G_1 with the fundamental group of the topological space W_1 consisting of the wedge of r_i n_i–dimensional tori for each i; similarly we let $G_2 = \pi_1(W_2)$, where W_2 is defined similarly using the s_i's. A finite cover of such a wedge of tori is homotopy equivalent to a wedge of tori.

We proceed in two steps. First, using finite covers we replace $(r_1, r_2, \dots, r_k)$ and $(s_1, s_2, \dots, s_k)$ by the sequences $(R_1, Y, Y, \dots, Y)$ and $(S_1, Y, Y, \dots, Y)$, respectively, where R_1, S_1, and Y are positive integers. Then we show that, again taking finite covers, we can leave the Y's unchanged and replace both R_1 and S_1 by a positive integer X, making the two sequences equal and completing the argument.

For the first step, let Y be a common multiple of $r_2, \dots, r_n, s_2, \dots s_n$. We construct W_1' and W_2' in the following way. W_1' is a finite cover of W_1 which for each $2 \leq i \leq k$ satisfies the following: it has Y tori of dimension n_i; each n_i–dimensional torus in W_1 has Y/r_i n_i–dimensional tori in W_1' which project to it; and each torus of W_1' covers its image in W_1 with degree r_i. Hence W_1' is a covering of W_1 with degree Y. Similarly construct W_2' from W_2 using the s_i. Note that W_1' and W_2' are each homotopy equivalent to wedges of tori (by contracting an embedded tree connecting the lifts of the basepoint), but this construction doesn't control the number of 1–dimensional tori in these wedges of tori.

For step 2 we notice that, given two spaces which are homotopy equivalent to wedges of tori of dimension up to n, if the number of n_i–dimensional tori is the same in each for all $2 \leq i \leq k$, then the number of 1–tori in the equivalent wedges

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of tori are equal if and only if the spaces have the same Euler characteristic. Now, note that a cyclic cover of degree d which is a connected cover on each n_i–torus for $i \geq 2$ leaves the number of tori of dimensions ≥ 2 unchanged. Thus, taking W_1' and W_2' as above, we can take a $\chi(W_2')$–fold cyclic cover of W_1' and a $\chi(W_1')$–fold cyclic cover of W_2' to obtain a pair of spaces homotopy equivalent to wedges of tori with the same number of tori for $2 \leq i \leq n$ and the same Euler characteristic. Hence these two covers are homotopy equivalent, thereby completing the argument.

(2) implies (3) for any finitely generated group. That (3) implies (1) follows from the easy fact that $\mathbb{Z}^n$ is quasi-isometric to $\mathbb{Z}^m$ if and only if $n = m$ plus the result of Papasoglu and Whyte [4]: given two groups which are not virtually $\mathbb{Z}$ and which admit graph of groups decompositions whose edge groups are finite and whose vertex groups have one or less ends, they are quasi-isometric if and only if they both have the same set of quasi-isometry types of vertex groups. □

The class of groups considered here is also quasi-isometrically rigid:

Theorem 2. *If a group G is quasi-isometric to a free product of finitely generated abelian groups then it is commensurable with such a free product.*

Proof. By the main theorem of [4] (Theorem 0.4), we have that G has a graph of groups decomposition with finite edge groups and where each vertex group is quasi-isometric to a finitely generated abelian group. Since a group is quasi-isometric to $\mathbb{Z}^i$ if and only if it has a finite-index subgroup isomorphic to $\mathbb{Z}^i$ [1, 2, 3], we claim that G has a finite-index subgroup which is a free product of finitely generated free abelian groups. This follows from the lemma below. □

Lemma. *If a group G splits as a finite graph of groups with finite edge groups and virtually torsion-free vertex groups G_v, then G has a finite-index subgroup which is a free product of a collection of torsion-free subgroups of each of the G_v's together with an additional free group.*

Proof. Let K_e denote the edge group corresponding to an edge e. A standard classifying space BG for G is obtained by gluing spaces into a "graph of spaces" with spaces BG_v at the vertices glued in the obvious way to spaces $BK_e \times [0, 1]$ for the edges. For each i let N_v be a torsion free normal subgoup of G_v of index y_v and let Y be a common multiple of the y_v. If e is an edge that abuts vertex i then the image of BK_e in BG_v lifts to $y_v/|K_e|$ copies of (the contractible space) $\widetilde{BK}_e$ in the cover BN_v of BG_v. Take a new graph of spaces where each BG_v is replaced by Y/y_v copies of BN_v and each BK_e is replaced by $Y/|K_e|$ copies of $\widetilde{BK}_e \times [0, 1]$. If e abuts vertex i we glue $y_v/|K_e|$ copies of the edge space to each of the Y/y_v BN_v's. We can clearly do this to get a space which is a connected finite cover of the original graph of spaces. It then has fundamental group a graph of groups with trivial edge groups and with vertex groups equal to copies of the N_v. This is a free product of the vertex groups and an additional free group given by the fundamental group of the underlying graph. □

We note that versions of the results will hold for free products allowing other groups that have many isomorphic subgroups of finite index and that are quasi-isometrically rigid up to commensurability, for example groups of the form (free)$\times\mathbb{Z}^n$ with $n \geq 1$, the Heisenberg group, etc.

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